

# Degree Distance of Adjoin of Trees

Padmapriya P. and Veena Mathad

**Abstract-** For a graph  $G = (V, E)$  the degree distance of  $G$  is defined as  $DD(G) = \sum_{u,v \in V(G)} (deg_G(u) + deg_G(v))d_G(u,v)$ ,

where  $deg_G(u)$  is the degree of the vertex  $u$  in  $G$  and  $d_G(u,v)$  is the shortest distance between  $u$  and  $v$ . In this paper we establish formulae to calculate the degree distance of adjoin of trees.

**Index Terms** - Tree graph, Star graph, Path graph, Wiener index, Degree distance.

## 1 INTRODUCTION

Topological indices are numbers associated with molecular graphs for the purpose of allowing quantitative structure activity/property/toxicity relationships. Wiener index is the best known topological index introduced by Harry Wiener to study boiling points of alkane molecules. By now, there exist a lot of different types of such indices which capture different aspects of the molecular graphs associated with the molecules under study. The degree distance index was introduced by Anderoy A. Dorynin and Amide A. Kochetova[3] and Gutman[4]. The degree distance is a weighted sum of the distances of vertices, it is also called the degree analogue of the Wiener index. Tomescu[8] showed that the star is the unique graph with minimum degree distance within the class of  $n$ -vertex connected graphs. Tomescu[9] deduced properties of graphs with minimum degree distance in the class of  $n$ -vertex connected graphs with  $m \geq n-1$  edges. The same result was obtained by Bucicovschi and Cioaba in [1]. Tomescu[10] determined the three smallest graphs with respect to degree distance among connected  $n$ -vertex graphs. Dankelmann[2] gave asymptotically sharp upper bounds for the degree distance. Also Ilic[5] calculated the degree distance of partial Hamming graphs.

For a graph  $G=(V,E)$  the degree distance of  $G$  is defined as

$$DD(G) = \sum_{u,v \in V(G)} (deg_G(u) + deg_G(v))d_G(u,v)$$

where  $deg_G(u)$  is the degree of the vertex  $u$  in  $G$  and  $d_G(u,v)$  is the shortest distance between  $u$  and  $v$ . Klein [1] showed that if  $G$  is a tree on  $n$  vertices then  $DD(G) = 4W(G) - n(n-1)$ , where

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v)$$

is the Wiener index of the graph  $G$ . Thus study of the degree distance for trees is equivalent to study of the Wiener index. But the degree distance of a graph is a more sensitive invariant than the Wiener index.

The degree distance of a vertex  $v$  in  $G$  is defined as

$$DD_G(v) = \sum_{u \in V(G)} (deg_G(u) + deg_G(v))d_G(u,v)$$

and the degree distance of pair of vertices  $u$  and  $v$  in  $G$  is defined as  $DD_G(u,v) = (deg_G(u) + deg_G(v))d_G(u,v)$ .

In this paper we obtain the formula to calculate the degree distance for some particular trees.

## 2 MAIN RESULTS

### 2.1 Star tree

Let  $E_m$  be star tree with  $m$  vertices  $v_0, v_1, \dots, v_{m-1}$  ( $m \geq 3$ ), shown in Fig.2.1

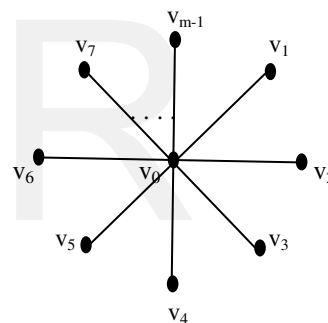


Fig.2.1: Star tree  $E_m$

#### Lemma 2.1

- (i)  $DD(E_m) = (m-1)(3m-4)$ ,
- (ii)  $DD_{E_m}(v_0) = m(m-1)$  and
- (iii)  $DD_{E_m}(v_i) = 5m-8$  for  $i = 1, 2, \dots, m-1$ .

Proof.

$$\begin{aligned} (i) DD(E_m) &= \sum_{u,v \in V(E_m)} (deg_u + deg_v)d(u,v) \\ &= [(m-1) + 1](m-1) + (m-2)(1+1)2 \\ &\quad + (m-3)(1+1)2 + \dots + 1(1+1)2 \\ &= (m-1)(3m-4). \end{aligned}$$

$$\begin{aligned} (ii) DD_{E_m}(v_0) &= \sum_{v_i \in V(E_m)} (deg_{v_i} + deg_{v_0})d(v_i, v_0) \\ &= \sum_{v_i \in V(E_m)} [1 + (m-1)].1 \\ &= m(m-1). \end{aligned}$$

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$$\begin{aligned}
 (iii) DD_{E_m}(v_i) &= \sum_{v_j \in V(E_m)} (degv_j + degv_i)d(v_j, v_i) \\
 &= \sum_{v_j \in V(E_m), v_j \neq v_0} (degv_j + degv_i)d(v_j, v_i) + [deg(v_0) + deg(v_i)]d(v_0, v_i) \\
 &= (m-2)(1+1)2 + [(m-1) + 1] \\
 &= 5m - 8.
 \end{aligned}$$

**2.2 Path tree**

Let  $P_m$  be Path Tree with  $m$  vertices  $v_0, v_1, \dots, v_{m-1}$  ( $m \geq 2$ ) shown in Fig.2.2

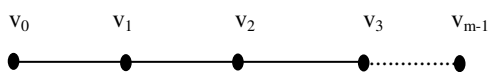


Fig.2.2: Path tree  $P_m$

**Lemma 2.2.**

$$(i) DD_{P_m}(v_0) = DD_{P_m}(v_{m-1}) = (m-1) \left[ 2 + \frac{3}{2}(m-2) \right]$$

$$(ii) DD(v_k, P_m) = 4k(k+1) + (2m-1)(m-1) - 4km$$

where  $k = 1, 2, \dots, m-2$ .

$$(iii) DD(P_m) = (m-1)(3m-4) + 8x$$

$$\text{where, } x = \begin{cases} \sum_{k=1}^{\frac{m-3}{2}} [m-k-2]k, & \text{if } m \text{ is odd;} \\ \sum_{k=1}^{\frac{m-4}{2}} [m-k-2]k, & \text{if } m \text{ is even.} \end{cases}$$

Proof.

$$\begin{aligned}
 DD_{P_m}(v_0) &= \sum_{v_j \in V(P_m)} (degv_j + degv_0)d(v_j, v_0) \\
 &= (degv_{m-1} + degv_0)d(v_{m-1}, v_0) + \sum_{v_j \neq v_{m-1}} (degv_j + degv_0)d(v_j, v_0) \\
 &= (1+1)(m-1) + (2+1)[1+2+3+\dots+(m-2)] \\
 &= (m-1) \left[ 2 + \frac{3}{2}(m-2) \right]
 \end{aligned}$$

$$\text{Similarly, } DD_{P_m}(v_{m-1}) = (m-1) \left[ 2 + \frac{3}{2}(m-2) \right]$$

$$\begin{aligned}
 DD_{P_m}(v_k) &= \sum_{v_j \in V(P_m)} (degv_j + degv_k)d(v_j, v_k) \\
 &= (degv_0 + degv_k)d(v_0, v_k) + (degv_{m-1} + degv_k)d(v_{m-1}, v_k) \\
 &+ \sum_{v_j \neq v_0, v_{m-1}} (degv_j + degv_k)d(v_j, v_k) \\
 &= 3k + 3[m-1-k] + 4\{[1+2+\dots+(k-1)] \\
 &+ [1+2+\dots+(m-2-k)]\}
 \end{aligned}$$

$$= 4k(k+1) + (2m-1)(m-1) - 4km.$$

$$\begin{aligned}
 DD(P_m) &= \sum_{u, v \in V(P_m)} (deg_u + deg_v)d_{P_m}(u, v) \\
 &= [(1+1)(m-1) + (2+1)(m-2) + (2+1)(m-3) + \dots + (2+1) \cdot 1] \\
 &+ (1+2)[1+2+\dots+(m-2)] + \{(2+2)[1+2+\dots+(m-3)] \\
 &+ (2+2)[1+2+\dots+(m-4)] + \dots + (2+2)[1]\} \\
 &= (m-1)(3m-4) + 4[(m-3)1 + (m-4)2 + \dots + 2(m-4) + 1(m-3)] \\
 &= (m-1)(3m-4) + 8 \begin{cases} \sum_{k=1}^{\frac{m-3}{2}} [m-k-2]k, & \text{if } m \text{ is odd;} \\ \sum_{k=1}^{\frac{m-4}{2}} [m-k-2]k, & \text{if } m \text{ is even.} \end{cases}
 \end{aligned}$$

**2.3 Adjoining tree with  $K_1$**

Let  $T_m$  is a tree adjoin with a vertex  $s \notin T_m$  as show in the Fig.2.3

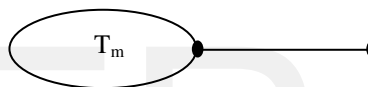


Fig.2.3:  $G = T_m \cdot K_1$

**Lemma 2.3.**

$$DD(T_m \cdot K_1) = DD(T_m) + W(s_0, T_m) + DD(s, G).$$

Proof.

$$\begin{aligned}
 DD(G) &= \sum_{u, v \in V(G)} (deg_u + deg_v)d(u, v) \\
 &= \sum_{u, v \in V(T_m)} (deg_u + deg_v)d(u, v) + \sum_{u \in V(T_m)} d(s_0, u) \\
 &+ \sum_{u \in V(G)} (deg_s + deg_u)d(s, u) \\
 &= DD(T_m) + W(s_0, T_m) + DD(s, G).
 \end{aligned}$$

**2.4 Adjoining nontrivial trees**

Let  $T_{m_1}$  and  $T_{m_2}$  be two trees that possess  $m_1$  and  $m_2$  vertices, respectively, adjoined at a vertex  $s$  (see Fig.2.4) and  $V^*(T_{m_i}) = V(T_{m_i}) \setminus \{s\}, i = 1, 2$ .

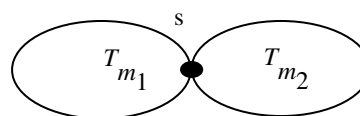


Fig.2.4: The tree  $T_{m_1} \cdot T_{m_2}$

**Lemma 2.4.**

$$DD(T_{m_1} \cdot T_{m_2}) = DD(T_{m_1}) - DD_{T_{m_1}}(s, T_{m_1}) + DD(T_{m_2}) - DD_{T_{m_2}}(s, T_{m_2}) + DD(s, T_{m_1} \cdot T_{m_2}) + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2})}} (deg_u + deg_v)d(u, v).$$

Proof.

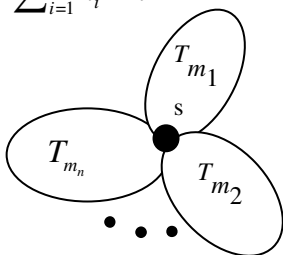
$$V^*(T_{m_i}) = V(T_{m_i}) \setminus \{s\}, i = 1, 2.$$

$$\begin{aligned} DD(T_{m_1} \cdot T_{m_2}) &= \sum_{u, v \in V(T_{m_1} \cdot T_{m_2})} (deg_u + deg_v)d(u, v) \\ &= \sum_{u, v \in V^*(T_{m_1})} (deg_u + deg_v)d(u, v) + \sum_{u, v \in V^*(T_{m_2})} (deg_u + deg_v)d(u, v) \\ &+ \sum_{u \in V(T_{m_1} \cdot T_{m_2})} (deg_u + deg_s)d(u, s) + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2})}} (deg_u + deg_v)d(u, v) \\ &= DD(T_{m_1}) - DD_{T_{m_1}}(s, T_{m_1}) + DD(T_{m_2}) - DD_{T_{m_2}}(s, T_{m_2}) \\ &+ DD(s, T_{m_1} \cdot T_{m_2}) + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2}), i < j}} (deg_u + deg_v)d(u, v). \end{aligned}$$

We generalize the above lemma as follows.

Let  $T_N$  be the tree formed by the trees  $T_N = T_{m_1} \cdot T_{m_2} \cdots T_{m_n}$  that possess  $m_1, m_2, \dots, m_n$  vertices, respectively, adjoined at a vertex  $s$ .  $V^*(T_{m_i}) = V(T_{m_i}) \setminus \{s\}, i = 1, 2, \dots, n$ .

$$N = \sum_{i=1}^n m_i - n + 1$$



**Fig.2.5:**  $T_N = T_{m_1} \cdot T_{m_2} \cdots T_{m_n}$

**Lemma 2.5.** Degree distance of the  $T_N$  is given by

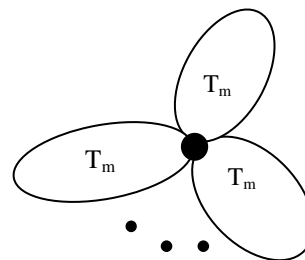
$$\begin{aligned} DD(T_N) &= \sum_{i=1}^n [DD(T_{m_i}) - DD_{T_{m_i}}(s, T_{m_i})] + DD(s, T_N) \\ &+ \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j})}} (deg_u + deg_v)d(u, v). \end{aligned}$$

Proof.

$$\begin{aligned} DD(T_N) &= \sum_{u, v \in V(T_N)} (deg_u + deg_v)d(u, v) \\ &= \sum_{u, v \in V^*(T_{m_1})} (deg_u + deg_v)d(u, v) + \sum_{u, v \in V^*(T_{m_2})} (deg_u + deg_v)d(u, v) \\ &+ \cdots + \sum_{u, v \in V^*(T_{m_n})} (deg_u + deg_v)d(u, v) + \sum_{u \in V^*(T_N)} (deg_u + deg_s)d(u, s) \\ &+ \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2})}} (deg_u + deg_v)d(u, v) + \sum_{\substack{u \in V^*(T_{m_2}), \\ v \in V^*(T_{m_3})}} (deg_u + deg_v)d(u, v) \\ &+ \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_3})}} (deg_u + deg_v)d(u, v) + \cdots + \sum_{\substack{u \in V^*(T_{m_{n-1}}), \\ v \in V^*(T_{m_n})}} (deg_u + deg_v)d(u, v) \\ &+ \cdots + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_n})}} (deg_u + deg_v)d(u, v) \\ &= \sum_{i=1}^n [DD(T_{m_i}) - DD_{T_{m_i}}(s, T_{m_i})] + DD(s, T_N) \\ &+ \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j})}} (deg_u + deg_v)d(u, v). \end{aligned}$$

If the trees  $T_{m_i}$  are similar graphs in the above lemma,

$T_{m_1} = T_{m_2} = \cdots = T_{m_n} = T_m$ ,  $DD(s, T_{m_i}) = DD(s, T_{m_j})$  for  $i, j \in \{1, 2, \dots, n\}$ ,  $N = nm - n + 1$



**Fig.2.6:**  $T_N = T_m \cdot T_m \cdots T_m$

**Corollary 2.6.** The degree distance of  $T_N$  is given by

$$\begin{aligned} DD(T_N) &= n[DD(T_m) - DD_{T_m}(s, T_m)] + DD(s, T_m) \\ &+ \frac{n(n-1)}{2} \sum_{\substack{u \in V^*(T_m), \\ v \in V^*(T_m)}} (deg_u + deg_v)d(u, v). \end{aligned}$$

Proof.

$$DD(T_N) = nDD(T_m) - nDD_{T_m}(s, T_m) + DD(s, T_m)$$

$$\begin{aligned}
 &+[1+2+\dots+(n-1)] \sum_{\substack{u \in V^*(T_m), \\ v \in V^*(T_m)}} (degu + degv)d(u, v) \\
 &= n[DD(T_m) - DD_{T_m}(s, T_m)] + DD(s, T_m) \\
 &+ \frac{n(n-1)}{2} \sum_{\substack{u \in V^*(T_m), \\ v \in V^*(T_m)}} (degu + degv)d(u, v).
 \end{aligned}$$

**2.5 Sequential adjoin of trees.**

Let  $T_N$  be the tree formed by trees  $T_{m_1}, T_{m_2}$  and  $T_{m_3}$  that possess  $m_1, m_2$  and  $m_3$  vertices, respectively, adjoined by two vertices  $s_1$  and  $s_2$ . The resulting tree is denoted by  $T_N = T_{m_1} - T_{m_2} - T_{m_3}$ , where  $N = m_1 + m_2 + m_3 - 2$  and  $V^*(T_{m_i}) = V(T_{m_i}) \setminus \{s_{i-1}, s_i\}, i = 1, 2, \dots, n$ .

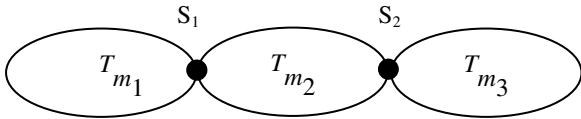


Fig.2.7

**Lemma 2.7.**

$$\begin{aligned}
 DD(T_N) &= \sum_{i=1}^3 \left[ DD(T_{m_i}) - \sum_{j=i-1}^i DD_{T_{m_i}}(s_j, T_{m_i}) \right] + \sum_{i=1}^2 DD(s_i, T_N) \\
 &- DD(s_1, s_2) + \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j}), i < j}} (degu + degv)d(u, v).
 \end{aligned}$$

Proof.

$$\begin{aligned}
 DD(T_N) &= \sum_{u, v \in V(T_N)} (degu + degv)d(u, v) \\
 &= \sum_{u, v \in V^*(T_{m_1})} (degu + degv)d(u, v) + \sum_{u, v \in V^*(T_{m_2})} (degu + degv)d(u, v) \\
 &+ \sum_{u, v \in V^*(T_{m_3})} (degu + degv)d(u, v) + \sum_{u \in V(T_N)} (degu + degs_1)d(u, s_1) \\
 &+ \sum_{u \in V(T_N)} (degu + degs_2)d(u, s_2) - (degs_1 + degs_2)d(s_1, s_2) \\
 &+ \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2})}} (degu + degv)d(u, v) + \sum_{\substack{u \in V^*(T_{m_2}), \\ v \in V^*(T_{m_3})}} (degu + degv)d(u, v) \\
 &+ \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_3})}} (degu + degv)d(u, v) \\
 &= \sum_{i=1}^3 \left[ DD(T_{m_i}) - \sum_{j=i-1}^i DD_{T_{m_i}}(s_j, T_{m_i}) \right] + \sum_{i=1}^2 DD(s_i, T_N)
 \end{aligned}$$

$$-DD(s_1, s_2) + \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j}), i < j}} (degu + degv)d(u, v).$$

We generalize the above lemma as follows

Let  $T_N$  be the tree formed by the trees  $T_{m_1}, T_{m_2}, \dots, T_{m_n}$  that possess  $m_1, m_2, \dots, m_n$  vertices, respectively, those trees are adjoined by the vertices  $s_1, s_2, \dots, s_{n-1}$ . We denote the tree  $T_{m_1} - T_{m_2} - \dots - T_{m_n}$  by  $T_N$  where  $N = \sum_{i=1}^n m_i - n + 1$ .

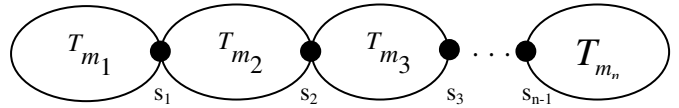


Fig.2.8:  $T_N = T_{m_1} - T_{m_2} - \dots - T_{m_n}$

**Lemma 2.8.** Degree distance of  $T_N$  is given by

$$\begin{aligned}
 DD(T_N) &= \sum_{i=1}^n \left[ DD(T_{m_i}) - \sum_{j=i-1}^i DD_{T_{m_i}}(s_j, T_{m_i}) \right] + \sum_{i=1}^{n-1} \left[ DD(s_i, T_N) - \sum_{j=1}^{i-1} DD(s_j, s_i) \right] \\
 &+ \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j}), i < j}} (degu + degv)d(u, v).
 \end{aligned}$$

Proof.

$$\begin{aligned}
 DD(T_N) &= \sum_{u, v \in V(T_N)} (degu + degv)d(u, v) \\
 &= \sum_{u, v \in V^*(T_{m_1})} (degu + degv)d(u, v) + \sum_{u, v \in V^*(T_{m_2})} (degu + degv)d(u, v) \\
 &+ \dots + \sum_{u, v \in V^*(T_{m_n})} (degu + degv)d(u, v) + \sum_{u \in V(T_N)} (degu + degs_1)d(u, s_1) \\
 &+ \sum_{u \in V(T_N)} (degu + degs_2)d(u, s_2) - DD(s_1, s_2) \\
 &+ \sum_{u \in V(T_N)} (degu + degs_3)d(u, s_3) - [DD(s_1, s_3) + DD(s_2, s_3)] \\
 &+ \dots + \sum_{u \in V(T_N)} (degu + degs_{n-1})d(u, s_{n-1}) \\
 &- [DD(s_1, s_{n-1}) + \dots + DD(s_{n-2}, s_{n-1})] + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_2})}} (degu + degv)d(u, v) \\
 &+ \sum_{\substack{u \in V^*(T_{m_2}), \\ v \in V^*(T_{m_3})}} (degu + degv)d(u, v) + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_3})}} (degu + degv)d(u, v) \\
 &+ \dots + \sum_{\substack{u \in V^*(T_{m_{n-1}}), \\ v \in V^*(T_{m_n})}} (degu + degv)d(u, v) + \dots + \sum_{\substack{u \in V^*(T_{m_1}), \\ v \in V^*(T_{m_n})}} (degu + degv)d(u, v) \\
 &= \sum_{i=1}^n \left[ DD(T_{m_i}) - \sum_{j=i-1}^i DD_{T_{m_i}}(s_j, T_{m_i}) \right] + \sum_{i=1}^{n-1} \left[ DD(s_i, T_N) - \sum_{j=1}^{i-1} DD(s_j, s_i) \right] \\
 &+ \sum_{\substack{u \in V^*(T_{m_i}), \\ v \in V^*(T_{m_j}), i < j}} (degu + degv)d(u, v).
 \end{aligned}$$

If  $T_{m_i}$  are similar tree graphs in the above lemma, then  $T_N$  is the tree graph  $T_m - T_m - \dots - T_m$  with  $N = nm - n + 1$  vertices.

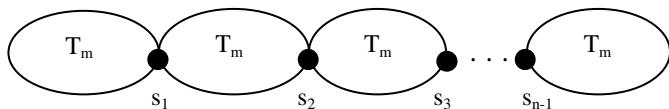


Fig.2.9.  $T_N = T_m - T_m - \dots - T_m$

**Corollary 2.9.**

$$DD(T_N) = nDD(T_m) - \sum_{i=1}^{n-1} [2DD(s_i, T_m) - DD(s_i, T_N)] - \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} DD(s_i, s_j)$$

$$+ \sum_{\substack{u \in V^*(T_m), \\ v \in V^*(T_m)}} (deg_u + deg_v)d(u, v).$$

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